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The representation and characterization of Drazin inverses of operators on a Hilbert space ☆

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Abstract

In this note, the representation and characterization of Drazin inverses of operators on a Hilbert space are established.

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1. Introduction

As we see in [2], the Drazin inverse has proved helpful in analyzing Markov chains, difference equation, differential equations and iterative procedures. It would be useful if these results could be extended to infinite dimensional situations. Applications could then be made to denumerable Markov chains, abstract Cauchy problems, infinite systems of linear differential equations, and possibly partial differential equations.

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In this note, we will consider the representation and characterization of Drazin inverses of operators on a Hilbert space. Let $\mathscr{H}$ and $\mathscr{K}$ be two complex Hilbert spaces. Denote by $\mathscr{B}(\mathscr{H}, \mathscr{K})$ the Banach space of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$. For an operator $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, the symbols $\mathscr{N}(T)$ and $\mathscr{R}(T)$ will denote the null space and the range space of T, respectively. The spectrum of T is denoted by $\sigma(T)$. Let $T \in \mathscr{B}(\mathscr{H})(= \mathscr{B}(\mathscr{H}, \mathscr{H}))$. If there exists an operator $X \in \mathscr{B}(\mathscr{H})$ satisfied the following four operator equations

$$\begin{cases} TXT = T, \\ XTX = X, \\ TX = (TX)^*, \\ XT = (XT)^*, \end{cases} \tag{1}$$

then X is called a Moore–Penrose inverse of T and denoted by T^+. As we know, T has a Moore–Penrose inverse if and only if $\mathscr{R}(T)$ is closed and the Moore–Penrose inverse of T is unique [1]. If there exists an operator $Y \in \mathscr{B}(\mathscr{H})$ satisfied the following three operator equations

$$\begin{cases} TY = YT, \\ YTY = Y, \\ T^{k+1}Y = T^k, \end{cases} \tag{2}$$

then Y is called a Drazin inverse of T and denoted by T^{D} [1]. Recall that $\operatorname{asc}(T)$ $(\operatorname{des}(T))$, the ascent (descent) of $T \in \mathscr{B}(\mathscr{H})$, is the smallest non-negative integer n such that $\mathscr{N}(T^n) = \mathscr{N}(T^{n+1})$ $(\mathscr{R}(T^n) = \mathscr{R}(T^{n+1}))$. If no such n exists, then $\operatorname{asc}(T) = \infty$ $(\operatorname{des}(T) = \infty)$. It is well known, $\operatorname{des}(T) = \operatorname{asc}(T)$ if $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ are finite [12,14]. An operator $T \in \mathscr{B}(\mathscr{H})$ has its Drazin inverse T^{D} if and only if it has finite ascent and descent, which is equivalent with that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$, say of order p. In such case $i(T) = \operatorname{asc}(T) = \operatorname{des}(T) = p$.

In recent years, the representation and characterization of the Drazin inverses of matrices or operators on a Hilbert space have been considered by many authors (see [1,2,4,7,8,10,11,14–17]). In this note, using the technique of block operator matrices and solving operator equations, a expression of the Drazin inverse of an operator on a Hilbert space are established and the characterization of the Drazin inverse of $n \times n$ matrices obtained by Zhang in [17] are extended through to operators in $\mathscr{B}(\mathscr{H})$. It is worthy to point out that the ideas and methods used in this note are different from that in [17].

2. Main results and proofs

We begin with some lemmas.

Lemma 2.1. *For an operator $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{R}(T)$ is closed if and only if there exists $X \in \mathscr{B}(\mathscr{H})$ such that $TXT = T$.*

Lemma 2.2. *Let $T \in \mathscr{B}(\mathscr{H})$. If there exist a nonnegative integer k and an operator $Y \in \mathscr{B}(\mathscr{H})$ such that $TY = YT$, $YTY = Y$ and $T^{k+1}Y = T^k$, then $\mathscr{R}(T^k) = \mathscr{R}(T^{k+1})$ and $\mathscr{R}(T^k)$ is closed.*

Proof. In general, $\mathscr{R}(T^{k+1}) \subseteq \mathscr{R}(T^k)$, but $T^{k+1}Y = T^k$ implies that $\mathscr{R}(T^k) \subseteq \mathscr{R}(T^{k+1})$. Hence $\mathscr{R}(T^k) = \mathscr{R}(T^{k+1})$.

Since $TY = YT$ and $YTY = Y$, $T^k = T^{k+1}Y = T^kYT = T^kYTYT = T^kY^2T^2 = \cdots = T^kY^kT^k$. By Lemma 2.1, $\mathscr{R}(T^k)$ is closed.

Lemma 2.3. *Let $T \in \mathscr{B}(\mathscr{H})$ and $i(T) = k$. If T^{D} is the Drazin inverse of T, then $\mathscr{R}(T^{\mathrm{D}}) = \mathscr{R}(T^k)$.*

Proof. From $T^{\mathrm{D}}TT^{\mathrm{D}} = T^{\mathrm{D}}$ and $TT^{\mathrm{D}} = T^{\mathrm{D}}T$, we have $T^{\mathrm{D}} = T^k(T^{\mathrm{D}})^{k+1}$. So $\mathscr{R}(T^{\mathrm{D}}) \subseteq \mathscr{R}(T^k)$. In the other hand, from $TT^{\mathrm{D}} = T^{\mathrm{D}}T$ and $T^{k+1}T^{\mathrm{D}} = T^k$, we can get $T^k = T^{\mathrm{D}}T^{k+1}$ and $\mathscr{R}(T^k) \subseteq \mathscr{R}(T^{\mathrm{D}})$. Therefore, $\mathscr{R}(T^{\mathrm{D}}) = \mathscr{R}(T^k)$.

Lemma 2.4 [1]. *Let $T \in \mathscr{B}(\mathscr{H})$. If the Drazin inverse T^{D} exists, then T^{D} is unique.*

If T has the Drazin inverse T^{D} and $i(T) = k$, then $\mathscr{R}(T^k)$ is an invariant subspace of T since $TT^k = T^{k+2}T^{\mathrm{D}} = T^kT^2T^{\mathrm{D}}$ [3]. *Therefore, T has the following operator matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \tag{3}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^k) \oplus \mathscr{R}(T^k)^{\perp}$.

Now, we shall discuss the representation of Drazin inverses.

Theorem 2.5. *Let $T \in \mathscr{B}(\mathscr{H})$ have the operator matrix form (3) and $i(T) = k$. If T has the Drazin inverse T^{D}, then*

$$T^{\mathrm{D}} = \begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \tag{4}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^k) \oplus \mathscr{R}(T^k)^{\perp}$.

Proof. If T is invertible, then $T^{\mathrm{D}} = T^{-1}$. The result holds. So we can assume that T is not invertible below.

If T has the Drazin inverse T^{D} and $i(T) = k$, by Lemma 2.3, $\mathscr{R}(T^{\mathrm{D}}) = \mathscr{R}(T^k)$. Then operator T^{D} have the operator matrix form

$$T^{\mathrm{D}} = \begin{pmatrix} T_{11}^{\times} & T_{12}^{\times} \\ 0 & 0 \end{pmatrix} \tag{5}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^k) \oplus \mathscr{R}(T^k)^{\perp} = \mathscr{R}(T^{\mathrm{D}}) \oplus \mathscr{N}((T^{\mathrm{D}})^{*})$.

From $T^{\mathrm{D}}TT^{\mathrm{D}} = T^{\mathrm{D}}$, we have

$$\begin{pmatrix} T_{11}^{\times}T_{11}T_{11}^{\times} & T_{11}^{\times}T_{11}T_{12}^{\times} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11}^{\times} & T_{12}^{\times} \\ 0 & 0 \end{pmatrix}.$$

Comparing the two sides of the above equation, we have $T_{11}^{\times}T_{11}T_{12}^{\times} = T_{12}^{\times}$ and $T_{11}^{\times}T_{11}T_{11}^{\times} = T_{11}^{\times}$. The first equation shows that $\mathscr{R}(T_{12}^{\times}) \subseteq \mathscr{R}(T_{11}^{\times})$ and $\mathscr{R}(T_{11}^{\times}) = \mathscr{R}(T_{11}^{\times}) + \mathscr{R}(T_{12}^{\times}) = \mathscr{R}(T^{\mathrm{D}})$. Moreover, the second equation shows that $T_{11}^{\times}T_{11} = I_{\mathscr{R}(T^{\mathrm{D}})}$, where I_K denotes the orthogonal projection on the closed subspace K.

From $TT^{\mathrm{D}} = T^{\mathrm{D}}T$, we have

$$\begin{pmatrix} T_{11}T_{11}^{\times} & T_{11}T_{12}^{\times} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11}^{\times}T_{11} & T_{11}^{\times}T_{12} + T_{12}^{\times}T_{22} \\ 0 & 0 \end{pmatrix}. \tag{6}$$

So $T_{11}T_{11}^{\times} = T_{11}^{\times}T_{11} = I_{\mathscr{R}(T^{\mathrm{D}})}$. This shows that T_{11} is invertible and $T_{11}^{-1} = T_{11}^{\times}$. In this case

$$TT^{\mathrm{D}} = \begin{pmatrix} T_{11}T_{11}^{\times} & T_{11}T_{12}^{\times} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{\mathscr{R}(T^{\mathrm{D}})} & T_{11}T_{12}^{\times} \\ 0 & 0 \end{pmatrix}.$$

From $T^{k+1}T^{\mathrm{D}} = T^k$, we have

$$\begin{pmatrix} T_{11}^{k} & T_{11}^{k+1}T_{12}^{\times} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11}^{k} & \sum_{i=0}^{k-1} T_{11}^{i}T_{12}T_{22}^{k-1-i} \\ 0 & T_{22}^{k} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have $T_{22}^k = 0$ and $T_{11}^{k+1}T_{12}^{\times} = \sum_{i=0}^{k-1} T_{11}^{i}T_{12}T_{22}^{k-1-i}$. So

$$T_{12}^{\times} = T_{11}^{-k-1}\sum_{i=0}^{k-1} T_{11}^{i}T_{12}T_{22}^{k-1-i} = \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i}.$$

Therefore,

$$T^{\mathrm{D}} = \begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix}.$$

Remark 2.6. By Theorem 2.5, a variety of the relations among Drazin inverses, Moore–Penrose inverses and {2}-inverses can be obtained.

(1) In general, an operator T and its Moore–Penrose inverse T^{+} are not commutative. And TT^{+} and $T^{+}T$ are orthogonal projections on $\mathscr{R}(T)$ and $\mathscr{R}(T^{*})$, respectively. But T and its Drazin inverse T^{D} are commutative and the product TT^{D} is an idempotent, in general, not an orthogonal projection. In this case,

$$TT^{\mathrm{D}} = \begin{pmatrix} I & \sum_{i=0}^{k-1} T_{11}^{i-k}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} = T^{\mathrm{D}}T.$$

(2) Recall that an operator $X \in \mathscr{B}(\mathscr{H})$ is said to be an {2}-inverse of $T \in \mathscr{B}(\mathscr{H})$ if $XTX = X$. For the details of the {2}-inverse, the reader can refer to [5,6,9,13,15]. It shows that the Drazin inverse T^{D} is an {2}-inverse of T if T^{D} exists.

In [9], for a pair $(\mathscr{M}, \mathscr{N})$ of closed subspaces, if $T|_{\mathscr{M}}$, the restriction of T on $\mathscr{M}$, is an invertible operator from $\mathscr{M}$ onto $T\mathscr{M}$ and $\mathscr{H} = T\mathscr{M} \oplus \mathscr{N}$, then there exists a unique $\{2\}$-inverse $T^{(2)}_{\mathscr{M}.\mathscr{N}}$ of T such that $\mathscr{R}(T^{(2)}_{\mathscr{M}.\mathscr{N}}) = \mathscr{M}$ and $\mathscr{N}(T^{(2)}_{M.N}) = N$. If the Drazin inverse T^{D} exists and $i(T) = k$, then, for the pair $(\mathscr{R}(T^k), \mathscr{N}(T^{*k}))$, we have

$$T^{(2)}_{\mathscr{R}(T^k),\mathscr{N}(T^{*k})} = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$\left(T^{\mathrm{D}} - T^{(2)}_{\mathscr{R}(T^k),\mathscr{N}(T^{*k})}\right)^2 = \begin{pmatrix} 0 & \sum_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix}^2 = 0$$

and

$$\begin{aligned} \left(T - \left(T^{(2)}_{\mathscr{R}(T^k),\mathscr{N}(T^{*k})}\right)^+\right)^{k+1} &= \left(\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} - \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}\right)^{k+1} \\ &= \begin{pmatrix} 0 & T_{12} T_{22}^k \\ 0 & T_{22}^{k+1} \end{pmatrix} \\ &= 0 \quad (\text{by } T_{22}^k = 0). \end{aligned}$$

(3) Denote $F = \sum_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i}$ and define an operator S by

$$S = \begin{pmatrix} I & T_{11}F \\ 0 & I \end{pmatrix},$$

then S is an invertible operator and

$$S^{-1} = \begin{pmatrix} I & -T_{11}F \\ 0 & I \end{pmatrix}.$$

Observing that

$$\begin{aligned} ST^{\mathrm{D}}S^{-1} &= \begin{pmatrix} I & T_{11}F \\ 0 & I \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & F \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -T_{11}F \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= T^{(2)}_{\mathscr{R}(T^k),\mathscr{N}(T^{*k})}. \end{aligned}$$

Let $\lambda \in \mathbb{C}$. Define λ^+ by

$$\lambda^+ = \begin{cases} \lambda^{-1}, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

For the spectrum $\sigma(T^{\mathrm{D}})$ of the Drazin inverse T^{D}, by Theorem 2.5, we have the following result.

Corollary 2.7. *Let $T \in \mathscr{B}(\mathscr{H})$. If the Drazin inverse T^{D} exists, then $\sigma(T^{\mathrm{D}}) = \{\lambda^+ : \lambda \in \sigma(T)\}$.*

Proof. If T is invertible, it has nothing to do. So we assume that T is non-invertible below.

If T has the Drazin inverse T^{D}, by the equations (3) and (4), $\sigma(T) = \sigma(T_{11}) \cup \{0\}$ and $\sigma(T^{\mathrm{D}}) = \sigma(T_{11}^{-1}) \cup \{0\}$ since $\{0\}$ is a set consisted of a simple point 0 , and $\sigma(T_{11}^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T_{11})\}$. Thus

$$\sigma(T^{\mathrm{D}}) = \{\lambda^+ : \lambda \in \sigma(T)\}.$$

Theorem 2.8. *Let $T \in \mathscr{B}(\mathscr{H})$ with $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$ and $i(T) = k$. Then there exists the unique operator $X \in \mathscr{B}(\mathscr{H})$ such that*

$$T^k X = 0, \quad XT^k = 0, \quad X^2 = X \quad \textit{and} \quad T^k + X \textit{ is invertible}.$$

Further, we have $X = I - TT^{\mathrm{D}}$.

Proof. If $T \in \mathscr{B}(\mathscr{H})$ with $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$ and $i(T) = k$, then $\mathscr{R}(T^k)$ is closed and invariant under T. Since $XT^k = 0$, the subspace $\mathscr{R}(T^k)$ is contained in $\mathscr{N}(X)$. Hence T and X have the following operator matrices

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & X_{12} \\ 0 & X_{22} \end{pmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^k) \oplus \mathscr{R}(T^k)^\perp$, respectively. Observing that

$$T^k = \begin{pmatrix} T_{11}^k & D \\ 0 & T_{22}^k \end{pmatrix},$$

where $D = \sum_{i=0}^{k-1} T_{11}^i T_{12} T_{22}^{k-1-i}$ and $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$. Then $\mathscr{R}(T_{22}^k) = \{0\}$, that is $T_{22}^k = 0$.

Since $T^k + X$ is invertible, and

$$T^k + X = \begin{pmatrix} T_{11}^k & D \\ 0 & T_{22}^k \end{pmatrix} + \begin{pmatrix} 0 & X_{12} \\ 0 & X_{22} \end{pmatrix} = \begin{pmatrix} T_{11}^k & D + X_{12} \\ 0 & X_{22} \end{pmatrix},$$

$\mathscr{R}(X_{22}) = \mathscr{R}(T^k)^\perp$. From the assumption that $X^2 = X$, we have $X_{22}^2 = X_{22}$. That is, $(X_{22} - I)X_{22} = 0$. Thus $X_{22} - I = 0$. Therefore T_{11}^k is invertible since $T^k + X$ is invertible. Moreover, T_{11} is invertible.

Since $T^k X = 0$, and

$$\begin{aligned} T^k X &= \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}^k \begin{pmatrix} 0 & X_{12} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} T_{11}^k & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & X_{12} \\ 0 & I \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & T_{11}^k X_{12} + D \\ 0 & 0 \end{pmatrix},$$

$T_{11}^k X_{12} + D = 0$. The invertibility of T_{11} implies that

$$X_{12} = -T_{11}^{-k} D = -T_{11}^{-k} \sum_{i=0}^{k-1} T_{11}^i T_{12} T_{22}^{k-1-i} = -\sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}.$$

Finally, we get

$$X = \begin{pmatrix} 0 & X_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -\sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i} \\ 0 & I \end{pmatrix}.$$

This shows that X uniquely exists.

Now we prove that $X = I - TT^{\mathrm{D}}$.

By Theorem 2.5, a directly computing shows that

$$\begin{aligned} I - TT^{\mathrm{D}} &= I - \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \\ &= I - \begin{pmatrix} I & \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i} \\ 0 & I \end{pmatrix} \\ &= X. \end{aligned}$$

Remark 2.9

(1) In Theorem 2.8, the condition that $T^k + X$ is invertible can been changed by $T^{\mathrm{D}} + X$ is invertible or $\mathscr{R}(T^k) + \mathscr{R}(X) = \mathscr{H}$.

(2) Since for each idempotent X there exists an invertible operator S such that SXS^{-1} is an orthogonal projection on $\mathscr{R}(X)$ and the Drazin inverse of an operator is similarly invariant, that is, if T^{D} is the Drazin inverse of T and S is an invertible operator, then $(STS^{-1})^{\mathrm{D}} = ST^{\mathrm{D}}S^{-1}$. Therefore, in the proof of Theorem 2.8, without loss of generality, we can assume that X is an orthogonal projection.

(3) Comparing this paper with [11,16], where the characterization of the Drazin inverse of $n \times n$ square matrix were established, we can see Theorem 2.8 covered the result of [11,16].

(4) Using a similar idea, it is not difficult to extend our results to the generalized Drazin inverse [4,8] of linear operator and W-weighted Drazin inverse [10,15] of linear operator.

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